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BASES FOR FD-MVD-STRUCTURES

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ABSTRACT

A characterization à la Armstrong, of dependency structures by subsets of the attribute set is studied for FD's, for MVD's and for FD's + MVD's.

We define a relation $B \succsim d$ (B agrees with d) between a subset B of the attribute set A and a dependency d, and show that the semantical deducibility of d from a set $\mathfrak{F}$ of dependencies, i.e. $\mathfrak{F} \models d$, is interpretable by : $\forall B \ (\forall d' \in \mathfrak{F} \ (B \succsim d') \Longrightarrow B \succsim d)$, which is denoted by $\mathfrak{F} \succeq d$. Besides the equivalence $(\mathfrak{F} \models d \Longleftrightarrow \mathfrak{F} \succeq d)$, we also show the equivalence $(\mathfrak{F} \succeq d \Longleftrightarrow \mathfrak{F} \vdash_{BFH} d)$, for every $\mathfrak{F}$ and d, where "$\vdash_{BFH}$" denotes the syntactical deducibility with respect to the axiom system given by Beeri, Fagin and Howard [BFH 77]. We propose finally the notion of a basis for $\mathfrak{F}$ and basis for a set of relations.

Our approach is related to that of Armstrong and Delobel [AD 80] but it is more general in the sense that it is applicable to mixed structures of FD's and MVD's. A comparison of the two approaches is given at the end of this paper.

RESUME

Une caractèrisation à la Armstrong, des structures de dépendance par des parties de l'ensemble d'attributs est étudiée pour les dépendances fonctionnelles (FD), pour les dépendances multi-valuées (MVD) et pour les dépendances fonctionnelles et multi-valuées (FD + MVD).

On définit une relation "$B \succ d$" (B accepte d) entre une partie B de l'ensemble fixe A d'attributs et une dépendance d sur A. Et on montre que la déductibilité sémantique de d à partir d'un ensemble $\mathfrak{F}$ de dépendances, $\mathfrak{F} \models d$, est interprétable par :

$$\forall B \in \mathcal{P}(A) \ (\forall d' \in \mathfrak{F} \ (B \succ d') \Longrightarrow B \succ d).$$

Pour cette formule, on propose la notation "$\mathfrak{F} \succ d$".

A partir de cette équivalence ($\mathfrak{F} \models d \Leftrightarrow \mathfrak{F} \succ d$), nous l'équivalence ($\mathfrak{F} \succ d \Leftrightarrow \mathfrak{F} \vdash_{BFH} d$), où "$\vdash_{BFH}$" denote la déductibilité syntaxique par rapport au système d'axiomes donné par Berri, Fagin et Howard [BFH 77]. On définit enfin la notion de base pour $\mathfrak{F}$ et de base pour un ensemble de relations.

Notre étude est liée à celle de Armstrong et Delobel [AD 80] mais elle est plus générale dans le sens qu'elle est applicable aux structures mixtes de dépendances fonctionnelles et multi-valuées. Une comparaison de ces deux méthodes est donné à la fin de ce papier.

BASIC NOTIONS

We assume that the universe U of attributes is given and that for each attribute $x \in U$, its domain D_x is also given. Intuitively, attributes are just variables and for every attribute x, its domain D_x is the set of the permitted values of x. Naturally we assume that the cardinality of D_x is not less than two, i.e. card $D_x \geq 2$.

An attribute set is a finite subset of U. We use the letter A to denote a fixed attribute set and the letters B, X, Y, Z, W, V to denote subsets of A.

Let D_A be the cartesian product of D_x's with $x \in A$, i.e. $D_A =_{def} \otimes_{x \in A} D_x$. A relation on A is a finite subset of D_A. We denote by $\mathfrak{R}_A$ the set of all relations on A, i.e. $\mathfrak{R}_A =_{def} \{R \subseteq D_A \mid \text{card } R \text{ is finite}\}$.

When a relation R is a relation on A, we say that A is the attribute set of R and denote it by $Att(R)$.

An element of D_A is called a tuple on A. Any $R \in \mathfrak{R}_A$ is therefore a finite set of tuples on A. For any tuple f of D_A and for any subset X of A, we denote by $f[X]$ or by $\Pi_X(f)$, the subtuple of f, which is an element of D_X. It is called the projection of f on X. The notion of projection is extended in the natural way to the relations $R \in \mathfrak{R}_A$ and to the subsets $\mathfrak{R}$ of $\mathfrak{R}_A$; i.e

$$R[X] =_{def} \{f[X] \mid f \in R\}$$

$$\mathfrak{R}[X] =_{def} \{R[X] \mid R \in \mathfrak{R}\}$$

Let $\underline{R} = \{R_i \mid i \in I\}$ be a finite set of relations, and let $\underline{A} = \{A_i \mid i \in I\}$ be the set of attribute sets such that $A_i = Att(R_i)$. The join (or natural join) of $\underline{R}$, denoted by $\bowtie \underline{R}$ or by $\bowtie_{i \in I} R_i$, is then a relation on the attribute set $A = \cup \underline{A}$, defined as follows :

$$\bowtie \underline{R} =_{def} \{f \in D_A \mid f[A_i] \in R_i \text{ for all } R_i \in \underline{R}\}$$

We shall denote by Att($\underline{R}$) the attribute set of the relation $\bowtie\underline{R}$; i.e. Att($\underline{R}$) = Att($\bowtie\underline{R}$) = $\cup\ \underline{A}$.

We are interested in those subsets of $\mathfrak{R}_A$ which are determined by certain constraints. Here we take as constraints the functional dependencies and the multivalued dependencies.

The set of all <u>functional dependencies</u> (abbr. FD's) on A is denoted by FD(A). We define it as the set of all the expressions of the form $X \to Y$ with $X, Y \subseteq A$. For any $R \in \mathfrak{R}_A$ and for any $d \in FD(A)$, the satisfaction relation "$R \mid\!\sim d$" (R <u>satisfies</u> d) is defined as follows :

$$R \mid\!\sim X \to Y \iff_{def} \forall f, g \in R(f[X] = g[X] \Rightarrow f[Y] = g[Y])$$

We extend the satisfaction relation to the set level as follows :

$$R \mid\!\sim \mathfrak{F} \iff_{def} \forall d \in \mathfrak{F}\ (R \mid\!\sim d)$$

$$\mathfrak{R} \mid\!\sim d \iff_{def} \forall R \in \mathfrak{F}\ (R \mid\!\sim d)$$

$$\mathfrak{R} \mid\!\sim \mathfrak{F} \iff_{def} \forall R \in \mathfrak{R}\ \forall d \in \mathfrak{F}\ (R \mid\!\sim d)$$

where $\mathfrak{R}$ is an arbitrary subset of $\mathfrak{R}_A$ and $\mathfrak{F}$ is an arbitrary subset of FD(A).

Based on the satisfaction relation, we define the following two functions (going on opposite directions) :

$$\mathcal{R} < A;FD > (\mathfrak{F}) =_{def} \{R \in \mathfrak{R}_A \mid R \mid\!\sim \mathfrak{F}\}$$

$$\mathcal{R}^{-1} < A;FD > (\mathfrak{R}) =_{def} \{d \in FD(A) \mid \mathfrak{R} \mid\!\sim d\}$$

with $\mathfrak{F} \subseteq FD(A)$ and $\mathfrak{R} \subseteq \mathfrak{R}_A$. These functions are functions between $\mathfrak{P}(FD(A))$ and $\mathfrak{P}(\mathfrak{R}_A)$, which are lattices with respect to set inclusion, and satisfy the following form of duality :

$$\mathfrak{R} \subseteq \mathfrak{R}_{< A;FD >}(\mathfrak{F}) \iff \mathfrak{F} \subseteq \mathfrak{R}^{-1}{}_{< A;FD >}(\mathfrak{R}).$$

We note also that these function are both order inversing, that is

$$\mathfrak{F} \subseteq \mathfrak{F}' \Rightarrow \mathfrak{R}(\mathfrak{F}) \supseteq \quad (\mathfrak{F}'),\ \mathfrak{R} \subseteq \mathfrak{R}' \Rightarrow \mathfrak{R}^{-1}(\mathfrak{R}) \supseteq \mathfrak{R}^{-1}(\mathfrak{R}')$$

and possess the following property :

$$\mathfrak{F} \subseteq \mathfrak{R}^{-1} \circ \mathfrak{R}(\mathfrak{F}),\ \mathfrak{R} \subseteq \mathfrak{R} \circ \mathfrak{R}^{-1}(\mathfrak{R}).$$

As $\mathfrak{R}^{-1} \circ \mathfrak{R}(\mathfrak{F})$ is greater than or equal to $\mathfrak{F}$, we may ask whether $d \in \mathfrak{R}^{-1} \circ \mathfrak{R}(\mathfrak{F})$ or not, for a given $\mathfrak{F} \subseteq FD(A)$ and a given $d \in FD(A)$. This is the <u>semantical deducibility</u> denoted by " $\mathfrak{F} \models d$", i.e.

$$\mathfrak{F} \models d \iff_{def} d \in \mathfrak{R}^{-1} \circ \mathfrak{R}(\mathfrak{F}).$$

A purely syntactical axiomatization is studied in [Arm. 74] and [BFH 77]. Generally, for every axiom system α, the syntactical deducibility "$\mathfrak{F} \vdash_{\alpha} d$" is compared with the semantical deducibility "$\mathfrak{F} \models d$". The syntactical deducibility "$\mathfrak{F} \vdash_{\alpha} d$" means that we can obtain from $\mathfrak{F}$ a set containing d after a <u>finite</u> number of successive applications of derivation rules of the axiom system α. Naturally an axiom system α should satisfy the following to make sense :

$$\forall\ \mathfrak{F} \subseteq FD(A) \quad \forall\ d \in FD(A) \quad (\mathfrak{F} \vdash_{\alpha} d \Rightarrow \mathfrak{F} \models d)$$

This property is called <u>soundness</u> of α with respect to the semantical deducibility "$\models$". It garantees that the axiom system is not too powerful. Another requirement for the axiom system is that it be sufficiently powerful. That is, we would like the following :

$$\forall\ \mathfrak{F} \subseteq FD(A) \quad \forall\ d \in FD(A) \quad (\mathfrak{F} \models d \Rightarrow \mathfrak{F} \vdash_{\alpha} d)$$

This is called <u>completeness</u> of the axiom system α with respect to the semantical deducibility "$\models$". Generally speaking, it may happen that no complete axiom system exists. But fortunately this is not the case with the functional dependencies, provided the cardinality of the domain is sufficiently large. Defining a sound and complete axiomatization is one way of characterizing the relation "$\models$".

We propose here a different characterization of "$\models$", based on an idea found in [Arm.74]. It amounts to defining another deducibility "$\supset=$" which is equivalent to the deducibility "$\models$".

In the same way as "$\models$" is defined based on the relation "$\mid\!\sim$", we define the relation "$\supset=$" based on the relation "$\supset\!\sim$".

The relation "$\supset\!\sim$" relates a subset of an attribute set A and a functional dependency on A. For any $B \subseteq A$ and any $d \in FD(A)$, "$B \supset\!\sim d$" is read as "B <u>agrees with</u> d" and it is formally defined as follows :

$$B \supset\!\sim X \to Y \quad \Longleftrightarrow_{\mathrm{def}} \quad (X \subseteq B \Rightarrow Y \subseteq B)$$

We can extend this relation to the set level, i.e. $B \supset\!\sim \mathfrak{F}$, $\mathfrak{B} \supset\!\sim d$, $\mathfrak{B} \supset\!\sim \mathfrak{F}$ with $\mathfrak{B} \subseteq \mathfrak{P}(A)$ and $\mathfrak{F} \subseteq FD(A)$, in a way similar to the one used in the case of "$\mid\!\sim$". The definition of the pair of set functions between the two lattices $\mathfrak{P}(FD(A))$ and $\mathfrak{P}(\mathfrak{P}(A))$ is also similar. Namely :

$$\mathfrak{B} < A;FD > (\mathfrak{F}) =_{\mathrm{def}} \{B \in \mathfrak{P}(A) \mid B \supset\!\sim \mathfrak{F}\}$$

$$\mathfrak{B}^{-1} < A;FD > (\mathfrak{B}) =_{\mathrm{def}} \{d \in FD(A) \mid \mathfrak{B} \supset\!\sim d\}$$

where $\mathfrak{F}$ is any subset of FD(A) and $\mathfrak{B}$ any subset of $\mathfrak{P}(A)$. We note also the duality and the other properties on this pair, as we have seen for pair $\mathfrak{R}$ and $\mathfrak{R}^{-1}$.

Now the definition of "$\supset=$" can be given as follows :

$$\mathfrak{F} \supset= d \Longleftrightarrow_{\text{def}} \quad d \in \mathfrak{B}^{-1} \circ \mathfrak{B}(\mathfrak{F})$$

Our claim is then :

$$\forall\, \mathfrak{F} \subseteq FD(A) \quad \forall\, d \in FD(A) \quad (\mathfrak{F} \models d \Longleftrightarrow \mathfrak{F} \supset= d)$$

In other words,

$$\forall\, \mathfrak{F} \subseteq FD(A) \quad (\mathfrak{R}^{-1} \circ \mathfrak{R}(\mathfrak{F}) = \mathfrak{B}^{-1} \circ \mathfrak{B}(\mathfrak{F}))$$

The importance of this relation might be clear, when we note that our mappings are as the following schema :

$$\begin{tikzcd}
\mathfrak{P}(\mathfrak{R}_A) \arrow[r, shift right, "\mathfrak{R}^{-1} < A;FD >"'] & \mathfrak{P}(FD(A)) \arrow[l, shift right, "\mathfrak{R} < A;FD >"'] \arrow[r, shift left, "\mathfrak{B} < A;FD >"] & \mathfrak{P}(\mathfrak{P}(A)) \arrow[l, shift left, "\mathfrak{B}^{-1} < A;FD >"]
\end{tikzcd}$$

Our discussion can be applied to other types of dependencies as well. For that purpose, it suffices to specify the set $*D(A)$ of all the dependencies concerned and the notions $R \mid\!\sim d$ and $B \supset\!\sim d$ for $R \in \mathfrak{R}_A$, $B \in \mathfrak{P}(A)$ and $d \in *D(A)$.

The set of all <u>Multi-valued dependencies</u> (abbr. MVD's) on A is denoted by MVD(A). It is defined as the set of all expressions of the form $X \twoheadrightarrow Y(A)$ with $X, Y \subseteq A$. When there is no missunderstanding, we abbreviate $X \twoheadrightarrow Y(A)$ by $X \twoheadrightarrow Y$.

For any $R \in \mathfrak{R}_A$ and any $d \in MVD(A)$, the satisfaction relation "$R \mid\!\sim d$" is defined as follows :

$$R \mid\!\sim X \twoheadrightarrow Y$$

$$\Longleftrightarrow_{\text{def}} \forall\, f,g \in R(f[X]=g[X] \Rightarrow \exists\, h \in R(h[X \cup Y] = f[X \cup Y] \wedge h[X \cup (A-Y)]=g[X \cup (A-Y)]))$$

We see immediately that

$$R \mid\!\sim X \twoheadrightarrow Y \quad \Longleftrightarrow \quad R = \bowtie\{R[X \cup Y], R[X \cup (A-Y)]\}$$

The corresponding agreement relation "$B \supset\!\sim d$", with $B \in \mathfrak{P}(A)$ and $d \in MVD(A)$, is defined as follows :

$$B \supset\!\sim X \twoheadrightarrow Y \Longleftrightarrow_{def} (X \subseteq B \Rightarrow Y \subseteq B \vee (A-Y) \subseteq B)$$

Now we may suppose that the two pairs of functions, $(\mathfrak{R} < A;{*D} >, \mathfrak{R}^{-1} < A;{*D} >)$ and $(\mathfrak{B} < A;{*D} >, \mathfrak{B}^{-1} < A;{*D} >$ are defined for $*D$ = FD and MVD. Moreover, we may even suppose it for the mixed case, say $*D$ = FMVD. That is, for example :

$$FMVD(A) =_{def} FD(A) \cup MVD(A)$$

$$\mathfrak{R} < A;FMVD > (\mathfrak{F}) =_{def} \mathfrak{R} < A;FD > (\mathfrak{F} \cap FD(A)) \cap \mathfrak{R} < A;MVD > (\mathfrak{F} \cap MVD(A))$$

$$\mathfrak{R}^{-1} < A;FMVD > (\mathfrak{R}) =_{def} \mathfrak{R}^{-1} < A;FD > (\mathfrak{R}) \cup \mathfrak{R}^{-1} < A;MVD > (\mathfrak{R})$$

As to the notions of deducibilities $\models$ and $\supset=$, we need no distinction, because if $\mathfrak{F} \cup \{d\} \subseteq {*D}(A)$, then $(\mathfrak{F} \models_{*D} d \Leftrightarrow \mathfrak{F} \models_{FMVD} d)$ and $(\mathfrak{F} \supset=_{*D} d \Leftrightarrow \mathfrak{F} \supset=_{FMVD} d)$, for $*D$ = FD or MVD.

AN ABSTRACT TREATMENT

Each of the above defined pairs of functions can be regarded as a pair of functions between a pair of lattices. In this section we study them in such a perspective.

Let A_1 and A_2 be a pair of finite lattices with partial order $\preceq$ and let Q_1, Q_2 be a pair of mappings from A_2 to A_1 and from A_1 to A_2, respectively (we shall use the same symbol for the partial orders of the two lattices, although they are in general different).

For our purposes it will be useful to impose the following conditions on Q_1 and Q_2. The nature of these conditions is suggested by the cases $*D$ = FD or MVD or FMVD.

Cond. 1 (Sub-element-duality)

$$x \preceq Q_1(y) \Longleftrightarrow y \preceq Q_2(x)$$

for any $x \in A_1$ and any $y \in A_2$.

Cond. 2.1

Q_1, Q_2 are order-inverting mappings

Cond. 2.2

$$x \prec Q_i \circ Q_j(x)$$

for any $x \in A_i$, with $(i,j) = (1,2)$ or $(2,1)$.

We remark then :

Proposition 1

Cond.1 is equivalent to (Cond. 2.1 + Cond. 2.2) □

Proof

=>) As $Q_j(x) \prec Q_j(x)$, we have Cond. 2.2 from Cond. 1.
Now we shall show Cond. 2.1 by using Cond. 2.2 and Cond. 1.
Let $x \prec x'$. Then by Cond. 2.2, $x \prec Q_i \circ Q_j(x')$. This is equivalent to $Q_j(x') \prec Q_j(x)$ by Cond. 1.

<=) Let $x \prec Q_j(y)$. Then by Cond. 2.1, $Q_i \circ Q_j(y) \prec Q_i(x)$.
But by Cond. 2.2, $y \prec Q_i \circ Q_j(y)$. So we have $y \prec Q_i(x)$. □

In what follows we assume that the pair of mappings Q_1, Q_2 satisfies Cond. 1, and therefore also satisfies Cond. 2.1 and Cond. 2.2.

We begin by remarking some fundamental properties.

Lemma 1.

$$Q_j \circ Q_i \circ Q_j(x) = Q_j(x)$$

for any $x \in A_i$ with $(i,j) = (1,2)$ or $(2,1)$ □

Proof

By Cond. 2.2, we have $Q_j(x) \prec Q_j \circ Q_i \circ Q_j(x)$ and $x \prec Q_i \circ Q_j(x)$ for any $x \in A_i$. And the latter relation implies $Q_j \circ Q_i \circ Q_j(x) \prec Q_j(x)$ by Cond. 2.1. □

As A_1, A_2 are lattices, the elements $x \cup x'$ and $x \cap x'$ exist for any pair of elements x, x' of A_i $(i=1,2)$. In this respect we have the following.

Lemma 2.

$$Q_i(x \cup x') = Q_i(x) \cap Q_i(x')$$

for any $x, x' \in A_j$ with $(i,j) = (1,2)$ or $(2,1)$. □

Proof

We shall show $Q_i(x \cup x') \prec Q_i(x) \cap Q_i(x')$ and its inverse.

$\prec$) As $x \prec x \cup x'$ and $x' \prec x \cup x'$, we have $Q_i(x \cup x') \prec Q_i(x)$ and $Q_i(x \cup x') \prec Q_i(x')$ by Cond. 2.1. Therefore $Q_i(x \cup x') \prec Q_i(x) \cap Q_i(x')$ by the property of the operation $\cap$.

$\succ$) It suffices to prove that $y \prec Q_i(x)$ and $y \prec Q_i(x')$ together imply $y \prec Q_i(x \cup x')$ for any $y \in \tilde{A}_i$. Assume $y \prec Q_j(y)$ and $x' \prec Q_j(y)$. It follows that $x \cup x' \prec Q_j(y)$ by the properties of the operation $\cup$. But this is equivalent to $y \prec Q_i(x \cup x')$ by Cond. 1. □

By these **lemmas** we may characterize the image sets $Q_i(A_j) =_{def} \{x \in A_i \mid \exists y \in A_j (x=Q_i(y))\}$ in two manners : Proposition 2 and Proposition 3.

Let $I_{Q_i \circ Q_j}(A_i)$ be the subset of A_i consisting of all the elements invariant under the composed mapping $Q_i \circ Q_j$.
Then :

Proposition 2.

$$Q_i(A_j) = I_{Q_i} \circ Q_j(A_i)$$

for $(i,j) = (1,2)$ or $(2,1)$. □

Proof :

$\subseteq$) By Lemma 1.

$\supseteq$) Because $Q_j(x) \in A_j$. □

Let $x =_{Q_j} x'$ and $[x]_{Q_j}$ be respectively, the equivalence relation and the equivalence class defined on A_i as follows

$$x =_{Q_j} x' \iff_{\text{def}} Q_j(x) = Q_j(x')$$

$$[x]_{Q_j} =_{\text{def}} \{x' \in A_i \mid x' =_{Q_j} x\}$$

Then by Lemma 2 we can say the following.

Lemma 3.

Every equivalence class $[x]_{Q_j}$ is closed under the lattice operation $\cup$ on A_i. □

Proof

Suppose $x', x'' \in [x]_{Q_j}$. Then by definition of $[x]_{Q_j}$, $x' =_{Q_j} x$ and $x'' =_{Q_j} x$. By Lemma 2, $Q_j(x' \cup x'') = Q_j(x') \cap Q_j(x'')$, therefore $Q_j(x' \cup x'') = Q_j(x) \cap Q_j(x)$. Thus we have $x' \cup x'' =_{Q_j} x$, that is $x' \cup x'' \in [x]_{Q_j}$. □

Every equivalence class $[x]_{Q_j}$ is therefore a finite semilattice and has a unique maximal element, denoted by $\max_{Q_j}(x)$ and called "the Q_j-maximal element associated to x". If $y = \max_{Q_j}(x)$ for some $x \in A_i$ we say that y is Q_j-maximal.

Lemma 4.

$$\max_{Q_j}(x) = Q_i \circ Q_j(x)$$

for any $x \in A_i$ with $(i,j) = (1,2)$ or $(2,1)$. □

Proof

Let y be any element of $[x]_{Q_j}$. We have to show $y \prec Q_i \circ Q_j(x)$. By Cond. 1, this is equivalent to showing $Q_j(x) \prec Q_j(y)$. But this is true because $Q_j(y) = Q_j(x)$ by our assumption. □

Corollary

$$\forall\, x, x' \in A_i \;\; (x =_{Q_j} x' \iff \max_{Q_j}(x) = \max_{Q_j}(x'))$$

Proof

=>) Immediate by Lemma 4.

<=) $\max_{Q_j}(x) = \max_{Q_j}(x')$ implies $Q_j(\max_{Q_j}(x)) = Q_j(\max_{Q_j}(x'))$, which in turn implies $Q_j(x) = Q_j(x')$, by Lemma 4 and 1.

Let us denote by $\max_{Q_j}(A_i)$ the set of all the Q_j-maximal elements of A_i. Then :

Proposition 3

$$Q_i(A_j) = \max_{Q_j}(A_i) \quad \text{for } (i,j) = (1,2) \text{ or } (2,1). \qquad \Box$$

Proof

$\supseteq$) Trivial by Lemma 4.

$\subseteq$) Let $x = Q_i(y)$ with $y \in A_j$. It suffices to show that $x = \max_{Q_j}(x)$. But this reduces to $x = Q_i \circ Q_j(x)$, by Lemma 4, and this is true by Lemma 1. □

Corollary

$$Q_i(A_j) = Q_i \circ Q_j(A_i) \quad \text{for } (i,j) = (1,2) \text{ or } (2,1). \quad \square$$

We end this section by remarking the following.

Proposition 4

$Q_i(A_j)$ is closed under the lattice operation $\cap$ on A_i, and so are therefore, $I_{Q_i \circ Q_j}(A_i)$, $\max_{Q_j}(A_i)$ and $Q_i \circ Q_j(A_i)$. $\square$

Proposition 5

Let (Q_1,Q_2) and (Q_1,Q_2') be pairs of mappings between the same pair of lattices, both pairs of mappings satisfying Cond. 1. Then $Q_2 = Q_2'$. $\square$

Proof

By Cond. 2.2 applied to (Q_1,Q_2) we have $x \prec Q_1 \circ Q_2(x)$, so $Q_2(x) \prec Q_2'(x)$, by Cond. 1 applied to (Q_1,Q_2'). In the same manner we get $Q_2'(x) \prec Q_2(x)$. Thus we have $Q_2(x) = Q_2'(x)$ for all $x \in A_1$. $\square$

So, for a pair (Q_1,Q_2) satisfying Cond. 1, it is reasonable to denote Q_i by Q_j^{-1} or by Q_j^c for $(i,j) = (1,2)$ or $(2,1)$. And therefore, we can denote the pair (Q_1,Q_2) by $< Q_1 >$ or by $< Q_2 >$. Moreover we shall abbreviate $Q_i \circ Q_j(x)$ by $x^{<Q_i>}$ and also by $x^{<Q_j>}$, which should not cause any confusion.

INTERPRETABILITY OF $\models$ by $\supset=$

For every case of $*D$, we had the following schema of mappings,

$$\mathfrak{P}(\mathfrak{R}_A) \; \underset{\mathfrak{R}^{-1} < A;*D >}{\overset{\mathfrak{R}< A;*D >}{\rightleftarrows}} \; \mathfrak{P}(*D(A)) \; \underset{\mathfrak{B}^{-1} < A;*D >}{\overset{\mathfrak{B} < A;*D >}{\leftrightarrows}} \; \mathfrak{P}(\mathfrak{P}(A))$$

And we had two sorts of semantical deducibility, "$\models$" and "$\supset=$". For any $\mathfrak{F} \subseteq *D(A)$ and any $d \in *D(A)$, they were defined as follows :

$$\mathcal{F} \mid= d \quad <=>_{def} \quad d \in \mathfrak{R}^{-1} < A;*D > \circ\, \mathfrak{R} < A;*D > \quad (\mathcal{F})$$

$$\mathcal{F} \supset= d \quad <=>_{def} \quad d \in \mathfrak{B}^{-1} < A;*D > \circ\, \mathfrak{B} < A;*D > \quad (\mathcal{F})$$

The meaningfullness of these definitions lies on the fact that $\mathfrak{R}^{-1} \circ \mathfrak{R}(\mathcal{F})$ and $\mathfrak{B}^{-1} \circ \mathfrak{B}(\mathcal{F})$ are the unique maximal elements of the equivalence classes $[\mathcal{F}]_{\mathfrak{R}}$ and $[\mathcal{F}]_{\mathfrak{B}}$, respectively.

We intend to prove the following :

$$\forall\, \mathcal{F} \subseteq \mathrm{FMVD}(A) \quad \forall\, d \in \mathrm{FMVD}(A) \qquad (\mathcal{F} \mid= d \;<=>\; \mathcal{F} \supset= d).$$

Let $\mathcal{F}$ be any subset of $*D(A)$ and d be any element of $*D(A)$. $\mathcal{F} \mid= d$ is equivalent to $\{d\} \subseteq \mathfrak{R}^{-1} \circ \mathfrak{R}(\mathcal{F})$ by definition, and the latter, in turn, is equivalent to $\mathfrak{R}(\mathcal{F}) \subseteq \mathfrak{R}(\{d\})$ by the duality property. In the same way, we see that $\mathcal{F} \supset= d$ is equivalent to $\mathfrak{B}(\mathcal{F}) \subseteq \mathfrak{B}(\{d\})$.

So the statements $(\mathcal{F} \mid= d => \mathcal{F} \supset= d)$, $(\mathcal{F} \supset= d => \mathcal{F} \mid= d)$ are respectively equivalent to :

$$\mathfrak{B}(\mathcal{F}) \not\subseteq \mathfrak{B}(\{d\}) \;=>\; \mathfrak{R}(\mathcal{F}) \not\subseteq \mathfrak{R}(\{d\})$$

$$\mathfrak{R}(\mathcal{F}) \not\subseteq \mathfrak{R}(\{d\}) \;=>\; \mathfrak{B}(\mathcal{F}) \not\subseteq \mathfrak{B}(\{d\})$$

To prove the first, we associate to every $B \in \mathfrak{P}(A)$ a relation $R_B \in \mathfrak{R}_A$ such that $B \in (\mathfrak{B}(\mathcal{F}) - \mathfrak{B}(\{d\}))$ implies $R_B \in (\mathfrak{R}(\mathcal{F}) - \mathfrak{R}(\{d\}))$. For the second, our method is a little different : we associate to every $R \in \mathfrak{R}_A$ a subset $\underline{B}_R \subseteq \mathfrak{P}(A)$ such that $R \in (\mathfrak{R}(\mathcal{F}) - \mathfrak{R}(\{d\}))$ implies $\underline{B}_R \subseteq \mathfrak{B}(\mathcal{F})$ and $\underline{B}_R \not\subseteq \mathfrak{B}(\{d\})$.

We begin by defining R_B for a given $B \in \mathfrak{P}(A)$. Let us fix a pair of tuples f_o and $f_1 \in D(A)$ such that $f_o(x) \neq f_1(x)$ for all $x \in A$. As card $D_x \geq 2$ for all $x \in A$, such a pair exists. The choice may be arbitrary. We next define the tuple $\chi_B \in D(A)$ depending on B, which fulfills the role of characteristic function of B.

$$\chi_B(x) =_{def} \begin{cases} f_o(x) & \text{if } x \in B \\ f_1(x) & \text{otherwise} \end{cases}$$

The definition of R_B is simply the following.

$$R_B =_{def} \{f_o', \chi_B\}$$

Then we have the following.

<u>Lemma 1</u>.

For any $B \in \mathcal{P}(A)$ and any $d \in FMVD(A)$,

$$B \supset\!\sim d \iff R_B \mid\!\sim d \qquad \Box$$

<u>Proof</u>

The case of $d \in FD(A)$ is trivial. Let us consider the case of $d \in MVD(A)$:

<=) Assume $B \not\supset\!\sim X \twoheadrightarrow Y$, that is, $X \subseteq B$, $Y \not\subseteq B$ and $A-Y \not\subseteq B$. We must find a tuple $f \in D(A)$ such that $f \notin R_B$ but $f \in \bowtie\{R_B[X \cup Y], R_B[X \cup (A-Y)]\}$. We simply have to define a tuple f as follows

$$f(x) = \begin{cases} \chi_B(x) & \text{if } x \in Y \\ f_o(x) & \text{if } x \in A-Y \end{cases}$$

In fact : $f \notin R_B$, that is $f \neq f_o$ and $f \neq \chi_B$. $f \neq f_o$, because $f[Y-B] = \chi_B[Y-B] = f_1[Y-B]$ and $Y-B \neq \emptyset$ by $Y \not\subseteq B$. $f \neq \chi_B$, because $f[A-Y-B] = f_o[A-Y-B]$, $\chi_B[A-Y-B] = f_1[A-Y-B]$ and $A-Y-B \neq \emptyset$ by $A-Y \not\subseteq B$.

It remains to show that $f[X \cup Y] \in R_B[X \cup Y]$ and $f[X \cup (A-Y)] \in R_B[X \cup (A-Y)]$. For that, we show $f[X \cup Y] = \chi_B[X \cup Y]$ and $f[X \cup (A-Y)] = f_o[X \cup (A-Y)]$. But this is clear from the definition of f, if we observe that $x \subseteq B$ implies $\chi_B[X] = f_o[X]$.

=>) Assume $\underline{B} \supset\!\sim X \twoheadrightarrow Y$, that is, $X \subseteq B \Rightarrow Y \subseteq B \vee A-Y \subseteq B$. If $\chi_B[X] \neq f_o[X]$, then $R_B \mid\!\sim X \twoheadrightarrow Y$. If $\chi_B[X] = f_o[X]$, then $X \subseteq B$, therefore $Y \subseteq B$ or $A-Y \subseteq B$. If $Y \subseteq B$, then $\chi_B[Y] = f_o[Y]$. If $A-Y \subseteq B$, then $\chi_B[A-Y] = f_o[A - Y]$. In every case , $R_B \mid\!\sim X \twoheadrightarrow Y$. □

Corollary

For any $\mathfrak{F} \subseteq FMVD(A)$ and any $d \in FMVD(A)$,

$$\mathfrak{F} \models d \quad \Rightarrow \quad \mathfrak{F} \supset\!= d. \qquad \square$$

Now we have to look for how to define the set $\underline{B}_R \subseteq \mathfrak{P}(A)$ for a given $R \in \mathfrak{R}_A$, so that $R \mid\!\sim d \Leftrightarrow \underline{B}_R \supset\!\sim d$ for any $d \in {*D}(A)$. We do it using a subset $FUSE(R) \subseteq (\mathfrak{P}(A) - \{A\})$, defined as follows :

For any pair of tuples $f,g \in D(A)$,

$$FUSE(f,g) =_{def} \{x \in A \mid f(x) = g(x)\}.$$

And for any $R \in \mathfrak{R}_A$,

$$FUSE(R) =_{def} \{FUSE(f,g) \mid f,g \in R \text{ and } f \neq g\}.$$

By definition, it is clear that

$$\forall\, d \in FD(A) \quad (FUSE(R) \supset\!\sim d \quad \Leftrightarrow \quad R \mid\!\sim d).$$

So, for the case of $*D = FD$, we have only to put $\underline{B}_R = FUSE(R)$. But this is not valid for the cases of $*D = MVD$ or $*D = FMVD$.

We are therefore interested in how far we can vary the set $\underline{B}_R$ when we have only FD's. And we remark that $\underline{B} \supset\!\sim X \to Y \Leftrightarrow \underline{B} \cup \{\cap\underline{B}\} \supset\!\sim X \to Y$ for any set $\underline{B} \subseteq \mathfrak{P}(A)$.

Let us define the closure of a set $\mathfrak{B} \subseteq \mathfrak{P}(A)$ with respect to the intersection of its elements.

$$\mathfrak{B}^{\cap} =_{def} \{\cap \underline{B} \mid \underline{B} \subseteq \mathfrak{B}\} .$$

In particular, if $\mathfrak{B} = \emptyset$, then $\mathfrak{B}^{\cap} = \{A\}$.

Let us also define the subset of $FUSE(R)^{\cap}$ consisting of the redundant elements with respect to the agreement of FD's.

$$REDUND(R) =_{def} \{B \in FUSE(R)^{\cap} \mid B \in (FUSE(R)^{\cap} - \{B\})^{\cap}\}$$

Naturally, $(FUSE(R)^{\cap} - REDUND(R)) \subseteq FUSE(R)$. And we define,

$$FUSE(R)^{-} =_{def} FUSE(R)^{\cap} - REDUND(R) .$$

Now we conclude from the above the following.

Lemma 2

$$\forall \mathfrak{B} (FUSE(R)^{-} \subseteq \mathfrak{B} \subseteq FUSE(R)^{\cap}$$

$$\Rightarrow \forall d \in FD(A) \ (\mathfrak{B} \mathrel{\supset\!\sim} d \Leftrightarrow R \mid\!\sim d) . \qquad \square$$

It is among these $\mathfrak{B}$'s lying between $FUSE(R)^{-}$ and $FUSE(R)^{\cap}$, that we hope to find our $\underline{B}_R$ satisfying

$$\forall d \in MVD(A) \ (\underline{B}_R \mathrel{\supset\!\sim} d \Leftrightarrow R \mid\!\sim d) .$$

As the following lemma shows, $FUSE(R)^{\cap}$ agrees essentially with no MVD's.

Lemma 3.

For any $X \twoheadrightarrow Y \in MVD(A)$,

$$FUSE(R)^{\cap} \mathrel{\supset\!\sim} X \twoheadrightarrow Y \Leftrightarrow FUSE(R)^{\cap} \mathrel{\supset\!\sim} X \rightarrow Y \text{ or } FUSE(R)^{\cap} \mathrel{\supset\!\sim} X \rightarrow A\text{-}Y. \quad \square$$

Proof

<=) Trivial

=>) Let $FUSE(R)^{\cap} \supset\!\not\sim X \to Y$ and $FUSE(R)^{\cap} \supset\!\not\sim X \to A-Y$. Then there exist B_1, B_2 such that $X \subseteq B_1, B_2$, $Y \not\subseteq B_1$ and $A-Y \not\subseteq B_2$. But $B_1, B_2 \in FUSE(R)^{\cap}$ implies $B_1 \cap B_2 \in FUSE(R)^{\cap}$, and $B_1 \cap B_2 \supset\!\not\sim X \to Y$. That is, $FUSE(R)^{\cap} \supset\!\not\sim X \twoheadrightarrow Y$. □

On the other hand, $FUSE(R)^{-}$ is, in general, too small. In fact, we may easily imagine the case where $FUSE(R) = \{B_1, B_2, B_1 \cap B_2\}$ with $B_1 \cup B_2 = A$ and $R \mid\!\not\sim B_1 \cap B_2 \twoheadrightarrow B_1$. Then $FUSE(R)^{-} = \{B_1, B_2\}$ and consequently $FUSE(R)^{-} \supset\!\sim B_1 \cap B_2 \twoheadrightarrow B_1$. In such a case the element $B_1 \cap B_2$ of REDUND(R) must not be rejected from our $\underline{B}_R$.

So we have to determine the part of REDUND(R) to be really rejected. We define it by :

$$REJECT(R) =_{def} \{B \in REDUND(R) \mid \exists\ B_1, B_2 \in FUSE(R)^{\cap}\ (B_1 \cap B_2 = B \wedge B_1 \cup B_2 = A \wedge B \subsetneq B_1 \wedge B \subsetneq B_2 \wedge R \mid\!\sim B \twoheadrightarrow B_1)\}.$$

And we define finally our $\underline{B}_R$ by :

$$\underline{B}_R =_{def} FUSE(R)^{\cap} - REJECT(R).$$

Lemma 4.

For any $d \in MVD(A)$

$\underline{B}_R \supset\!\sim d \iff R \mid\!\sim d.$ □

Proof

<=) We assume $R \mid\!\sim X \twoheadrightarrow Y$. And we examine every possibility of $B \in FUSE(R)^{\cap}$ satisfying $B \supset\!\not\sim X \twoheadrightarrow Y$ to conclude that there is no $B \in \underline{B}_R$ such that $B \supset\!\not\sim X \twoheadrightarrow Y$.

We shall use the following fact : if $B \in FUSE(R)$, then $R \mid\!\sim B \twoheadrightarrow Y$ implies $B \cup Y \in FUSE(R)$ and $B \cup (A-Y) \in FUSE(R)$.

We first suppose that $B \supset\!\!\not\sim X \twoheadrightarrow Y$ for some $B \in FUSE(R)$. Then by $R \mid\!\sim X \twoheadrightarrow Y$, we have $R \mid\!\sim B \twoheadrightarrow Y$, because $B \supset\!\!\not\sim X \twoheadrightarrow Y$ implies $X \subseteq B$. As $B \in FUSE(R)$, it follows that $B \cup Y \in FUSE(R)$ and $B \cup (A-Y) \in FUSE(R)$. And they are not equal to B, because $B \supset\!\!\not\sim X \twoheadrightarrow Y$ implies $Y \not\subseteq B$, $A-Y \not\subseteq B$. Thus, we conclude that such B must satisfy $B \in REDUND(R)$.

We next suppose that $B \supset\!\!\not\sim X \twoheadrightarrow Y$ for some $B \in REDUND(R)$ (B may or may not be in $FUSE(R)$). Then B can be represented as $B = \cap \underline{B}'$ with some $\underline{B}' \subseteq FUSE(R)$, where $\underline{B}' \subseteq FUSE(R)$ instead of $\underline{B}' \subseteq FUSE(R)^{\cap}$, because every element of $FUSE(R)^{\cap} - FUSE(R)$ is representable as the intersection of some elements of $FUSE(R)$. And for every $B' \in \underline{B}'$, $R \mid\!\sim B' \twoheadrightarrow Y$, because $X \subseteq B'$ by $B \supset\!\!\not\sim X \twoheadrightarrow Y$ and $B \subseteq B'$. So, as we have noted above, $B' \cup Y \in FUSE(R)$ and $B' \cup (A-Y) \in FUSE(R)$ for every $B' \in \underline{B}'$. It follows that $B_1 = B \cup Y$ and $B_2 = B \cup (1-Y)$ are both elements of $FUSE(R)^{\cap}$. And $B \subsetneq B_1$, $B \subsetneq B_2$ by $B \supset\!\!\not\sim X \twoheadrightarrow Y$. That is, $B \in REJECT(R)$.

Thus we have shown that under the assumption $R \mid\!\sim X \twoheadrightarrow Y$, no element B such that $B \supset\!\!\not\sim X \twoheadrightarrow Y$, can be found in $FUSE(R)^{\cap} - REJECT(R)$, i.e. in $\underline{B}_R$.

=>) Let us assume that $R \mid\!\not\sim X \twoheadrightarrow Y$ and show that there exists in $\underline{B}_R$ an element B such that $X \subseteq B$ and such that $R \mid\!\not\sim B \twoheadrightarrow Y$, from which will follow that $B \supset\!\!\not\sim X \twoheadrightarrow Y$.

We show it in two steps ;

Step 1. Showing that $\exists\, B \in FUSE(R)$ $(X \subseteq B \wedge R \mid\!\not\sim B \twoheadrightarrow Y)$.

Step 2. Showing that if $R \mid\!\not\sim B \twoheadrightarrow Y$ and $B \in REJECT(R)$, then $\exists\, B' \in FUSE(R)^{\cap}$ $(B \subsetneq B' \wedge R \mid\!\not\sim B' \twoheadrightarrow Y)$.

These two facts ensure us the existence of an increasing sequence of elements of $FUSE(R)^{\cap}$, terminating by an element which is not in REJECT(R), **and**, therefore, this element is in $\underline{B}_R$, because the cardinality of $FUSE(R)^{\cap}$ is finite.

Step 1) By $R \not\mid\!\sim X \twoheadrightarrow Y$, there exists a pair of tuples $f,g \in R$ such that $f[X] = g[X]$ for which no element h of R satisfies both $h[X \cup Y] = f[X \cup Y]$ and $h[X \cup (A-Y)] = g[X \cup (A-Y)]$. Let $B = FUSE(f,g)$. Then evidently $X \subseteq B$ and $R \not\mid\!\sim B \twoheadrightarrow Y$.

Step 2) Let us assume that $B \in REJECT(R)$. Then by definition there is a pair, $B_1, B_2 \in FUSE(R)^{\cap}$ such that $B_1 \cap B_2 = B$, $B_1 \cup B_2 = A$, $B \subsetneq B_1$, $B \subsetneq B_2$ and $R \mid\!\sim B \twoheadrightarrow B_1$. Evidently, the relation $R \mid\!\sim B \twoheadrightarrow B_2$, also holds. Suppose now that $R \mid\!\sim B_1 \twoheadrightarrow Y$ and $R \mid\!\sim B_2 \twoheadrightarrow Y$ are both true. Then we must have both $R \mid\!\sim B \twoheadrightarrow B_1 \cap Y$ and $R \mid\!\sim B \twoheadrightarrow B_2 \cap Y$. It follows that $R \mid\!\sim B \twoheadrightarrow Y$, because $B_1 \cup B_2 = A$. Thus we can conclude that if $R \not\mid\!\sim B \twoheadrightarrow Y$ for some $B \in REJECT(R)$, then there is $B' \in FUSE(R)^{\cap}$ such that $B \subsetneq B'$, for which $R \not\mid\!\sim B' \twoheadrightarrow Y$. □

We have obtained the following theorem.

<u>Theorem</u> .

For any $\mathfrak{F} \subseteq FMVD(A)$ and any $d \in FMVD(A)$,

$$\mathfrak{F} \models d \iff \mathfrak{F} \supset\!= d. \qquad \square$$

<u>Proof</u>

=>) Corollary to Lemma 1.

<=) Lemma 2 and Lemma 4. □

AN APPLICATION

As an application of our result, let us consider the soundness and the completeness of an axiom system, say α, of FMVD(A). Now, instead of proving $\mathfrak{F} \models d \Leftrightarrow \mathfrak{F} \vdash_{\alpha} d$, we may prove $\mathfrak{F} \supset= d \Leftrightarrow \mathfrak{F} \vdash_{\alpha} d$. As an example, let us take the axiom system of [BFH.77], say BFH.

Proof of the soundness of BFH.

We must prove that $\mathfrak{F} \vdash_{BFH} d \Rightarrow \mathfrak{F} \supset= d$. It may go through by induction on the minimum complexity of deduction of d from $\mathfrak{F}$ by BFH. And the problem reduces to interpreting the inference of every axiom by the logical acceptability of the corresponding inference on the agreement relation. For example, for the rule MVD 2, in [BFH 77], we verify whether $B \supset X \twoheadrightarrow Y$ implies $B \supset X \cup W \twoheadrightarrow Y \cup Z$, provided $Z \subseteq W$. These verifications present no particular difficulty. □

Proof of the completeness of BFH.

We must prove that $\mathfrak{F} \supset= d \Rightarrow \mathfrak{F} \vdash_{BFH} d$, for every $\mathfrak{F} \subseteq *D(A)$ and every $d \in *D(A)$. Only the case of $*D$ = FMVD is explicitly treated in the following but it can be read also as a proof of each restricted case, i.e. $*D$ = FD or $*D$ = MVD.

a. We first consider the case where d is an FD. Let us denote by $P(\mathfrak{F},X,Y)$ or simply by $P(X)$ the following statement :

$$\mathfrak{F} \supset= X \to Y \quad \Rightarrow \quad \mathfrak{F} \vdash_{BFH} X \to Y.$$

We suppose that $\mathfrak{F}$ and Y are arbitrarily fixed and prove $\forall X\, P(X)$ by induction on X. As P(A) is evidently true, we want to prove the following induction step :

$$\forall X' (X \subsetneq X' \Rightarrow P(X')) \Rightarrow P(X).$$

If $Y \subseteq X$, this step is also trivial. So we assume (1) $Y \nsubseteq X$; (2) $\forall X'(X \subsetneq X' \Rightarrow P(X'))$ and (3) $\mathfrak{F} \supset= X \to Y$, and we prove $\mathfrak{F} \vdash X \to Y$.

The condition (3) means that $\forall B \in \wp(A)$ $(B \supset\sim \mathfrak{F} \Rightarrow B \supset\sim X \to Y)$. And (1) means that $X \supset\not\sim X \to Y$. So we have $X \supset\not\sim \mathfrak{F}$.

<u>Case 1</u> : $X \supset\not\sim \mathfrak{F} \cap FD(A)$.

Let $Z \to W$ be an FD such that $Z \to W \in \mathfrak{F}$ and $X \supset\not\sim Z \to W$. Then we have (4) $\mathfrak{F} \vdash X \to W \cup X$ and (5) $P(W \cup X)$ by the axiom FD 2, in [BFH 77], and by our assumption (2), because the condition $X \supset\not\sim Z \to W$ implies $Z \cup X = X$ and $X \subsetneqq W \cup X$. But our assumption (3) implies $\mathfrak{F} \supset= W \cup X \to Y$, so, by (5) we have (6) $\mathfrak{F} \vdash W \cup X \to Y$. Thus we obtain our conclusion $\mathfrak{F} \vdash X \to Y$ by the axiom FD 3 with our (4) and (6).

<u>Case 2</u> : $X \supset\sim \mathfrak{F} \cap FD(A)$ and $X \supset\not\sim \mathfrak{F} \cap MVD(A)$.

Note that this is not the case when $*D \neq FMVD$.

Let $Z \twoheadrightarrow W$ be an MVD such that $Z \twoheadrightarrow W \in \mathfrak{F}$ and $X \supset\not\sim Z \twoheadrightarrow W$. Then we have

(4') $\mathfrak{F} \vdash X \twoheadrightarrow W$

(4") $\mathfrak{F} \vdash X \twoheadrightarrow (A-W)$,

(5') $P(W \cup X)$,

(5") $P((A-W) \cup X)$,

by the axioms MVD 2, MVD 0 of [BFH 77] and by our assumption (2), because the condition $X \supset\not\sim Z \twoheadrightarrow W$ implies $Z \cup X = X$, $X \subsetneqq W \cup X$ and $X \subsetneqq (A-W) \cup X$. But our assumption (3) implies that $\mathfrak{F} \supset= W \cup X \to Y$ and $\mathfrak{F} \supset= (A-W) \cup X \to Y$, so, by (5') and (5") we have

(6') $\mathfrak{F} \vdash W \cup X \to Y$,

(6") $\mathfrak{F} \vdash (A-W) \cup X \to Y$.

Now, by the axiom FD-MVD 3, of [BFH 77] we can derive

(7') $\mathfrak{F} \vdash X \rightarrow Y - W$,

(7") $\mathfrak{F} \vdash X \rightarrow Y - (A-W)$

from (4') and (6'), or from (4") and (6"), respectively. And by the axiom FD 5 of [BFH 77] we can finally conclude $\mathfrak{F} \vdash X \rightarrow Y$ from (7') and (7").

b. We next consider the case where d is an MVD. Let us denote by $Q(\mathfrak{F},X,Y)$ or simply by $Q(X)$ the following statement :

$$\mathfrak{F} \supset= X \twoheadrightarrow Y \Rightarrow \mathfrak{F} \vdash_{\overline{BFH}} X \twoheadrightarrow Y .$$

This time also, we suppose that $\mathfrak{F}$ and Y are arbitrarily fixed and prove $\forall X\ Q(X)$ by induction on X. Q(A) is trivial. To prove the induction step, we assume (1') $X \not\supset= X \twoheadrightarrow Y$ (2') $\forall X'(X \subsetneq X' \Rightarrow Q(X'))$ and (3') $\mathfrak{F} \supset= X \twoheadrightarrow Y$. Then as we have seen above, we arive to the condition $X \not\supset= \mathfrak{F}$, and we devide it in two cases.

<u>Case 1</u> : $X \not\supset= \mathfrak{F} \cap MVD(A)$.

The proof proceeds as in Case 2 of part a. This time we use MVD 4 and MVD 5 in stead of FD-MVD 3 and FD 5 (see [BFH 77])

<u>Case 2</u> : $X \supset\sim \mathfrak{F} \cap MVD(A)$ and $X \supset\sim \mathfrak{F} \cap FD(A)$.

Note that this is not the case when $*D \neq FMVD$.

Let $Z \rightarrow W$ be an FD such that $Z \rightarrow W \in \mathfrak{F}$ and $X \not\supset= Z \rightarrow W$. The condition $X \not\supset= Z \rightarrow W$ implies $Z \cup X = X$ and $X \subsetneq W \cup X$. If $X \subsetneq (A-W) \cup X$, then we can proceed with the proof as in Case 1, because we may use the condition $Q((A-W) \cup X)$ otherwise, $A - W \subseteq X$ is true and so is $Y - W \subseteq X$; therefore $\mathfrak{F} \vdash X \rightarrow W$ implies $\mathfrak{F} \vdash X \rightarrow Y$ by FD 2 [BFH 77] . □

CONCLUSION

For an FD $d=X \to Y$, the definition of $B \supset\!\sim d$ is quite natural. But, for an MVD $d = X \twoheadrightarrow Y$, it is not so evident, because the notion of MVD is not a simple generalization on the logical complexity of the notion of FD, such as $X \to Y \vee (A-Y)$ for example.

This fact is reflected on the part of the proof concerning the soundness of "$\supset\!\!=$" ($\mathfrak{F} \supset\!= d \Rightarrow \mathfrak{F} \models d$); in fact, for *D = FD, the set $\underline{B}_R$ such that $\forall d \in {*D}(A)$ $(\underline{B}_R \supset\!\sim d \Leftrightarrow R \mid\!\sim d)$, can be defined simply by FUSE(R), whereas for *D = MVD and FMVD, the notion of FUSE(R) was not sufficient to define it.

In any way, our quite simple definition of the notion of $B \supset\!\sim X \twoheadrightarrow Y$ was good, and we could get the equation

$$\mathfrak{R}^{-1} < A;{*D} > \circ\, \mathfrak{R} < A;{*D} > (\mathfrak{F}) = \mathfrak{B}^{-1} < A;{*D} > \circ\, \mathfrak{B} < A;{*D} > (\quad)$$

not only for *D = MVD, besides *D = FD, but also for *D = FMVD ; that is, the definitions of $\supset\!\sim$ for FD and for MVD are found compatible. Thanks to this equation, we can introduce the notion of basis as follows.

Let $\mathfrak{B} \subseteq \mathfrak{P}(A)$ and let $\mathfrak{F} \subseteq {*D}(A)$. Suppose

$$\mathfrak{B} = \mathfrak{B}^{-1} < A;{*D} > \circ\, \mathfrak{B} < A;{*D} > (\mathfrak{F})$$

Then it follows

$$\forall d \in {*D}(A) \quad (\mathfrak{B} \supset\!\sim d \Leftrightarrow \mathfrak{F} \models d),$$

and vice versa, by the above equation. In this case, $\mathfrak{B}$ characterizes perfectly the set $\mathfrak{F}$ with respect to the semantical deducibility of *D's. This justifies the following definition.

Definition

A subset $\mathfrak{B}$ of $\mathfrak{P}(A)$ is called a <u>basis</u> of a family $\mathfrak{F}$ of *D's on A, (with respect to *D), if

$$\mathfrak{B} = \mathfrak{B}^{-1}{}_{< A;*D >} \mathfrak{B}_{< A;*D >} (\mathfrak{F}) \qquad \square$$

In the same spirit, we introduce also the following.

Definition

A subset $\mathfrak{B}$ of $\mathfrak{P}$(A) is called a basis of a family of relations, $\mathfrak{R} \subseteq \mathfrak{R}_A$ (with respect to *D), if

$$\mathfrak{B} = \mathfrak{B}^{-1}{}_{< A;*D >} \mathfrak{B}_{< A;*D >} (\mathfrak{F})$$

and

$$\mathfrak{R} = \mathfrak{R}^{-1}{}_{< A;*D >} \mathfrak{R}_{< A;*D >} (\mathfrak{F})$$

for some $\mathfrak{F} \subseteq$ *D(A). $\square$

As every subset of $\mathfrak{P}$(A) is a basis of some $\mathfrak{F} \subseteq$ *D(A), we may call $\mathfrak{P}(\mathfrak{P}(A))$ the family of all the bases.

The impossibility of extending this study up to the case of *D = CMVD or FCMVD (where CMVD = contexted MVD), is discussed in a separate paper.

Note :

A large part of this work is already studied in [AD 80], where the notion of MVD is replaced by an equivalent notion, i.e. that of decomposition. In place of our $\mathfrak{B}$ < A;MVD > ($\mathfrak{F}$) with $\mathfrak{F}$ an arbitrary set of MVD's, they introduce the notion of antiroot to characterize a full family of decompositions.

If we replace the notion of decomposition by that of MVD, we may express the notion of antiroot as follows :

Let $\mathcal{F}^*$ be a full family of MVD's and let $\wp(A)_{\leq -2}$ be the family of those subsets of A whose complement set has cardinality ≥ 2, i.e.

$$\wp(A)_{\leq -2} =_{def} \{B \in \wp(A) \mid \text{card } B \leq \text{card } A - 2\}.$$

Then the set of antiroots of $\mathcal{F}^*$ is the following :

$$AR(\mathcal{F}^*) =_{def} \mathcal{B}< A;MVD > (\mathcal{F}^*) \cap \wp(A)_{\leq -2}$$

Concerning our theorem, their results can be expressed as follows in our terminology :

1. $\forall\, d \in MVD(A)\ (d \in \mathcal{F}^* \iff AR(\mathcal{F}^*) \succ d)$.

2. For any $R \in \mathcal{R}_A$, we have :

$$FUSE(R)^- \subseteq AR(\mathcal{R}^{-1} < A;MVD > (R)) \cup (FUSE(R)^{\cap})_{-1} \subseteq FUSE(R)^{\cap},$$

where

$$(FUSE(R)^{\cap})_{-1} =_{def} \{B \in FUSE(R)^{\cap} \mid \text{card } B = \text{card } A - 1\}.$$

3. Let the attribute set A contain at least two elements. Then for any full family $\mathcal{F}^*$ of MVD's on A, such that $\emptyset \in AR(\mathcal{F}^*)$ and for any $\mathcal{B}^1 \subseteq \wp(A)_{-1}$ with

$$\wp(A)_{-1} =_{def} \{B \in \wp(A) \mid \text{card } B = \text{card } A - 1\},$$

there exists $R \in \mathcal{R}_A$ such that :

$$\mathcal{R}^{-1} < A;MVD > (R) = \mathcal{F}^*$$

and

$$(AR(\mathcal{F}^*) \cup \mathcal{B}^1)^{\cap} = FUSE(R)^{\cap}.$$

The first and second results above should be compared with our Lemmas 2 and 4. The third result should be compared with our Lemma 1.

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